Sarcouncil Journal of Applied Sciences

ISSN(Online): 2945-3437

Volume- 05| Issue- 03| 2025

Research Article **Received:** 02-02-2024 | **Accepted:** 20-02-2025 | **Published:** 15-03-2025

Some Properties of Harmonic and Power M-Harmonic Functions

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Abstract: In this paper it is proved, if the $u(z)$ function of the unit ball is subharmonic and $u^m(z)$ $M-$harmonic, then $u^m(z)$ is pluriharmonic and if $U^2 \subset \mathbb{C}^2$ polidisce function is harmonic and $M-$harmonic, then it is n harmonic.

Keywords: holomorphic function, real-analytic function, harmonic function, $n-$harmonic function, subharmonic function, pluriharmonic function, $M-$harmonic function, plurisubharmonic function, $M-$subharmonic function.

INTRODUCTION

Pluriharmonic functions play an important role in the theory of functions of many complex variables. Being a real part of analytical functions, they are often used in studies of geometric properties of a class of analytical functions, in the theory of integral formulas, in complex potential theory, and others (Ронкин, Н. И 1971; Kim, H. O 1996; Furstenberg, H 1963; Stein, E. M., & Weiss N. J, 1969).

It is well known that a function $u \in C^2(D)$ is called harmonic (subharmonic) in the domain $D \subset \mathbb{C}^n$ (D a fixed domain in $\mathbb{C}^n$) if it satisfies the Laplace equation everywhere in D

$$\Delta u = 0 \quad (\Delta u \geq 0), \quad \text{где} \quad \Delta u = 4\sum_{i=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_i}.$$

Definition 1.1.

Function u is called harmonic n if it is harmonic for each variable individually. Function u must satisfy n equations $\Delta_j u = 0 \quad (j = 1,\dots,n)$.

Definition1.2.

Function $u \in C^2(D)$ is called pluriharmonic in D, if it satisfies the following n^2 differential equations

$$\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} = 0 \qquad (i, j = 1,\dots,n).$$

Suppose B is an open unit ball in $\mathbb{C}^n$ centered at the origin ($B = \{z \in \mathbb{C}^n : |z| < 1\}, \partial B = \{\omega \in \mathbb{C}^n : |\omega| = 1\}$), $\varphi_a(z)$ is a linear-fractional biholomorphic mapping of B onto itself of the following form:

$$\varphi_a(z) = \frac{a - P_a(z) - \left(1 - |a|^2\right)^{\frac{1}{2}}\left(z - P_a(z)\right)}{1 - \langle z, a\rangle}.$$

For $a \in B$, $a \neq 0$, $\varphi_a(0) = a$, $\varphi_a(a) = 0$, $\varphi_0(z) = -z$, where the angle brackets denote the Hermitian scalar product of $\langle z, w\rangle = z_1\bar{w}_1 + z_2\bar{w}_2 + z_3\bar{w}_3 + \dots z_n\bar{w}_n = \sum_{i=1}^{n} z_i\bar{w}_i$, $|z| = \langle z, z\rangle^{\frac{1}{2}}$, and $P_a(z) = \frac{\langle z, a\rangle}{\langle a, a\rangle}a$, $a \neq 0$, $P_0(z) = 0$. Obviously, $\varphi_a(0) = a$, $\varphi_a(a) = 0$.

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DOI- https://doi.org/10.5281/zenodo.15028212

Sarc. Jr. Appl. Sci. vol-5, issue-3 (2025) pp-1-7

The invariant Laplace operator $\tilde{\Delta}$ of functions on twice smooth in B is defined as follows:

$$\tilde{\Delta} f(z) = \Delta(f \circ \varphi_z)(0)$$

Where $z \in B$, Δ is the usual Laplacian

The operator $\tilde{\Delta}$ is called invariant because it commutes with automorphisms of the ball B in the following sense (see [4] pp. 54-66): $\tilde{\Delta}(f \circ \varphi) = (\tilde{\Delta} f) \circ \varphi$, where $f \in C^2(G)$ is ($G \subset B$), and φ is any biholomorphic automorphism of the ball B.

If $G = \{z \in \mathbb{C}^n : \langle z, a \rangle \neq 1\}$ then φ_a holomorphically maps G into $\mathbb{C}^n$. It is clear that $G \supset \bar{B}$, since $|a| < 1$.

A function $f(z)$ that is twice continuously differentiable on an open set $D \subset B$ is called M-harmonic [4] in D if for each z from D the equality $\tilde{\Delta} f = 0$ $\left(\tilde{\Delta} f \geq 0\right)$ holds. It is also known (see (Рудин, У 1984) pp. 54-66) that

$$\tilde{\Delta} f = 4(1 - |z|^2) \sum_{j,k=1}^{n} \left(\delta_{jk} - z_j \bar{z}_k\right) \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} \quad ,$$

where $\delta_{jk} = 0$ for $j \neq k$, $\delta_{jj} = 1$.

If $f(z)$ is an M-harmonic function, then (by Theorem 4.2.5 (see [4] pp. 54-66)) it is real analytic in B, therefore $\Delta f(z)$ is also real analytic and, therefore, $\Delta f(z) = 0$ is everywhere on B.

The theorems of Hartogs and Lelon state that separately holomorphic or separate harmonic functions are, respectively, holomorphic or harmonic functions in a set of variables. The radial analogue of Hartogs' main theorem was proved by F. Forelli (1978) and R. M. Madrakhimov (1986).

A theorem similar to Hartogs' theorem was proved by F. Forelli in (Рудин, У 1984).

Theorem 1.1.
Let a B-open unit ball in C^n and a function $f(z): B \to R$($f(z): B \to C$) satisfy the following conditions:

1) $f \in \cap \{C^k(0):\ k = 1,2,\ldots\}$;
2) for each fixed $z \in B$, the function $f(\lambda z)$ is harmonic ($f(\lambda z)$is holomorphic) in the circle $\{\lambda \in C, \lambda z \in B\}$.

Then the function $f(z)$ is pluriharmonic (holomorphic) in B.

The following theorem was proven in the work of R. Madrakhimov (Мадрахимов Р.М 2008).

Theorem 1.2.
Let a B-open unit ball in C^n and a function $f(z): B \to R$ $(f(z): B \to C)$ satisfy the following conditions:
1) $f(z) \in sh(0)$ $\left(\mathrm{Re} f(z) \in sh(0)\right)$;
2) for each fixed $z \in B$, the function $f(\lambda z)$ is harmonic ($f(\lambda z)$ is holomorphic) in the disk $\{\lambda \in C,\ \lambda z \in B\}$.
Then the function $f(z)$ is pluriharmonic (holomorphic) in B.

The following theorem was proved by W. Rudin in (Rudin, W 1977): if a function f(z) is harmonic and M-harmonic in B, then f(z) is a pluriharmonic function in B.
For the polycircle $U^n \subset \mathbb{C}^n$ a similar definition holds.

Definition 1.5.
Let U^n be the unit polydisk in C^n. The function $f(z) \in C^2(U^n)$ is called M-harmonic (M-subharmonic) in U^n if $\tilde{\Delta} f(z) = 0$ $(\tilde{\Delta} f(z) \geq 0)$ is in ^{n}, where $\tilde{\Delta} f(z) = 2 \sum_{j=1}^{n} \left(1 - |z_j|^2\right)^2 \frac{\partial^2 f(z)}{\partial z_j \partial \bar{z}_j}$ is the invariant Laplacian in U^n (see [4], pp. 24-25).

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Rudin in [3] obtained the following result: if the function $u(z)$ is harmonic and M is harmonic in B, then $u(z)$ is pluriharmonic in B.

In [7] the following theorem was proven: if the function $u(z)$ is harmonic and M is subharmonic in B, then $u(z)$ is pluriharmonic in B.

In [8] it was proven that if the function $u(z)$ is subharmonic and M is harmonic in B, then $u(z)$ is pluriharmonic in B.

MAIN RESULTS OF THE WORK

Theorem 2.1.

Let B be the unit ball in $\mathbb{C}^n$ and $u(z)$ and satisfy the following conditions:

a) the function $u(z)$ is harmonic in B;

b) the function $u^m(z)$ is M-harmonic in B.

Then $u^m(z)$ is pluriharmonic in B.

Proof of Theorem 2.1.

Since the M-harmonic function $u^m(z)$ on the unit ball is real-analytic, it can be expanded into a series of homogeneous polynomials:

$$u^m(z) = \sum_{k=1}^{\infty} P_k(z) \qquad (1)$$

where $P_k(z)$ is a homogeneous polynomial $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$ of degree k. The series converges uniformly on compact subsets of the unit ball, and successive derivatives $u^m(z)$ can be obtained by differentiating with respect to the series. If in the given expansion $P_k(z)$ is the smallest nonzero homogeneous polynomial of degree k, then we say that the function $u^m(z)$ has general order k. If $u^m(z) \equiv 0$, meaning that all homogeneous terms in the expansion are zero, then the function $u^m(z)$ has general order ∞. The series converges uniformly on compact subsets of B, and successive derivatives of the function $u^m(z)$ can be obtained by differentiating this series.

Each polynomial in can be written as

$$P_k(z) = \sum_{p+q=k} f_{p,q}, \qquad (2)$$

where $f_{p,q}$ are polynomials in which the total order with respect to $z_1, \dots, z_n$ is p and the total order with respect to $\bar{z}_1, \dots, \bar{z}_n$ is q. Each term of the polynomial $f_{p,q}$ has the form $M(z) = C z_1^{\alpha_1} \dots z_n^{\alpha_n} \bar{z}_1^{\beta_n} \dots \bar{z}_n^{\beta_n}$, $\alpha_1 + \dots + \alpha_n = p, \beta_1 + \dots + \beta_n = q$. From the conditions of the theorem $\Delta u = 0$, $\Delta u^m(z) = (1-|z|^2)(\Delta u^m(z) - 4\sum_{i,j=1}^{n} z_i \bar{z}_j \frac{\partial^2 u^m(z)}{\partial z_i \partial \bar{z}_j}) = 0$ and

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$$\Delta u^m = 4\sum_{i=1}^{n} \frac{\partial^2 u^m}{\partial z_i \partial \overline{z}_i} =$$

$$= 4\sum_{i=1}^{n} m(m-1)u^{m-2} \frac{\partial u}{\partial z_i}\frac{\partial u}{\partial \overline{z}_i} + 4\sum_{i=1}^{n} mu^{m-1} \frac{\partial^2 u}{\partial z_i \partial \overline{z}_i} = 4m(m-1)u^{m-2}\sum_{i=1}^{n} \frac{\partial u}{\partial z_i}\overline{\frac{\partial u}{\partial z_i}} =$$

$$= 4m(m-1)u^{m-2}\sum_{i=1}^{n} \left|\frac{\partial u}{\partial z_i}\right|^2.$$

Let us prove the theorem for the following cases:

a) Let m be an even number. Then $u^{m-2}\sum_{i=1}^{n} \left|\frac{\partial u}{\partial z_i}\right|^2 \geq 0$. It is easy to see that $\Delta u^m = 4\sum_{i,j=1}^{n} z_i \overline{z}_j \frac{\partial^2 u^m}{\partial z_i \partial \overline{z}_j} \geq 0$. From this we obtain the inequality

$$\sum_{i,j=1}^{n} z_i \overline{z}_j \frac{\partial^2 u^m}{\partial z_i \partial \overline{z}_j} = \sum_{k=2}^{\infty} \sum_{p+q=k} pqf_{p,q} \geq 0.$$

Consequently, at $pq \neq 0$ all polynomials $f_{p,q}$ vanish, since in this case $\sum_{p+q=k} pqf_{p,q}(0) = 0$, and from the minimum principle we obtain that $\sum_{p+q=k} pqf_{p,q} \equiv 0$, it follows that $pqf_{p,q} = 0$ for all p and q. Hence $f_{pq} = 0$ or $p = 0$ or $q = 0$.

Hence, the expansion of $P_k(z)$ polynomial in (2.8) consists only of terms of the form $f_{p,0}$ and $f_{0,q}$, that is, $P_k(z)$ is a pluriharmonic polynomial and the sum of series (2), that is, $u^m(z)$ is a pluriharmonic function, and the theorem is proved.

b) Let m be an odd number, then $u^{m-2}\sum_{i=1}^{n} \left|\frac{\partial u}{\partial z_i}\right|^2 \geq 0$ or $u^{m-2}\sum_{i=1}^{n} \left|\frac{\partial u}{\partial z_i}\right|^2 \leq 0$. We have proved the case of $u^{m-2}\sum_{i=1}^{n} \left|\frac{\partial u}{\partial z_i}\right|^2 \geq 0$ in a). Let us prove it for the case of $u^{m-2}\sum_{i=1}^{n} \left|\frac{\partial u}{\partial z_i}\right|^2 \leq 0$. It was proved above that $\Delta u^m = 4m(m-1)u^{m-2}\sum_{i=1}^{n} \left|\frac{\partial u}{\partial z_i}\right|^2$. Therefore, $\Delta u^m = 4\sum_{i,j=1}^{n} z_i \overline{z}_j \frac{\partial^2 u^m}{\partial z_i \partial \overline{z}_j} \leq 0$. From this we get that

$$\sum_{i,j=1}^{n} z_i \overline{z}_j \frac{\partial^2 u^m}{\partial z_i \partial \overline{z}_j} = \sum_{k=2}^{\infty} \sum_{p+q=k} pqf_{p,q} \leq 0.$$

Consequently, at $pq \neq 0$ all polynomials $f_{p,q}$ vanish, since in this case $\sum_{p+q=k} pqf_{p,q}(0) = 0$, and from the maximum principle we obtain that $\sum_{p+q=k} pqf_{p,q} \equiv 0$, it follows that $pqf_{p,q} = 0$ for all p and q. Hence $f_{pq} = 0$ or $p = 0$ or $q = 0$.

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Hence, the expansion of $P_k(z)$ polynomial in (2.8) consists only of terms of the form $f_{p,0}$ and $f_{0,q}$, that is, $P_k(z)$ is a pluriharmonic polynomial and the sum of series (2), that is, $u^m(z)$ is a pluriharmonic function, and the theorem is proved. The theorem is proven.

Note:

If function $u(z)$ is harmonic in B, is function $u^2(z)$ harmonic?

The answer to this question is negative, as the following example of function $u(z) = z_1\bar{z}_1 + z_2\bar{z}_2$, $z \in B \subset \mathbb{C}^2$ shows.

Since $\Delta u = 0$ is in B, and $\Delta u^2 = 8(z_1\overline{z}_1 + z_2\overline{z}_2)$, then function $u^2(z)$ is not harmonic in $B \cap \{z_1 \neq 0 \text{ or } z_2 \neq 0\}$.

It is known that the ball and the polycircle are biholomorphically nonequivalent. If we consider a similar question in the polycircle, then we get a weaker result.

Recall from the introduction that a function f on U^n is n-harmonic (or strongly harmonic) if it is harmonic in each variable separately. Thus a C^2 function is n-harmonic on U^n if and only if $\tilde{\Delta}_j f = 0$ for all $j = 1, \ldots, n$. We now give an example of a function which satisfies $\tilde{\Delta}_{U^n} f = 0$, but which is not n-harmonic on U^n. For $z \in U, t \in T$, let

$$P(z,t) = \frac{(1-|z|^2)}{|1-z\bar{t}|^2}$$

denote the Poisson kernel on U. As we will see in Proposition 5.4, for fixed $t \in T, \alpha > 0$,

$$\tilde{\Delta}_U P^\alpha(z,t) = 2\alpha(\alpha-1)P^\alpha(z,t)$$

For $i = 1,2$, let $\alpha_i = \lambda_1 + \frac{1}{2}$. Then

$$\tilde{\Delta}_i P^{\frac{1}{2}+\lambda_i}(z_i,t_i) = 2\left(\lambda_i^2 - \frac{1}{4}\right)P^{\frac{1}{2}+\lambda_i}(z_i,t_i)$$

Thus if $F(z_1,z_2) = P^{\frac{1}{2}+\lambda_1}(z_1,t_1)P^{\frac{1}{2}+\lambda_2}(z_2,t_2)$,

$$\tilde{\Delta}_{U^2} F = 2\left(\lambda_1^2 + \lambda_2^2 - \frac{1}{2}\right)F$$

Hence for any λ_1, λ_2 satisfying $\lambda_1^2 + \lambda_2^2 = \frac{1}{2}, \tilde{\Delta}_{U^2} F = 0$. If λ_1 and λ_2 are not equal to $\frac{1}{2}$, the function F is not 2 -harmonic. Thus for an appropriate choice of λ_1, λ_2, F is weakly harmonic but not strongly harmonic.

It is quite clear that every n-harmonic function $u(z)$ in U^n satisfies $\Delta u = \tilde{\Delta} u = 0$. For $n = 2$ we will prove the opposite of this proposition.

Theorem 2.2. Let $U^2 \subset \mathbb{C}^2$ and function $u(z)$ satisfy the following conditions:

a) Function $u(z)$ is harmonic in U^2;

b) Function $u(z)$ is M-harmonic in U^2.

Then $u(z)$ is 2-harmonic in U^2.

Proof 2.2. If $u(z)$ is a harmonic and M-harmonic function in U^2, we obtain equalities $\frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2} = 0$ and $\left(1-|z_1|^2\right)^2 \frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1} + \left(1-|z_2|^2\right)^2 \frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2} = 0$. From the obtained equalities it follows that $\frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2} = -\frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1}$ and we arrive at the equation:

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$$\left(1-|z_1|^2\right)^2\frac{\partial u}{\partial z_1\partial \overline{z}_1}-\left(1-|z_2|^2\right)^2\frac{\partial u}{\partial z_1\partial \overline{z}_1}=\frac{\partial u}{\partial z_1\partial \overline{z}_1}\left(|z_2|^2-|z_1|^2\right)\left(2-|z_1|^2-|z_2|^2\right)=0 \quad (3).$$

From equation (3) follow the relations $\frac{\partial u}{\partial z_1\partial \overline{z}_1}=0$ or $|z_1|=|z_2|$.

Similarly, expressing the first term through the second, i.e.

$$\frac{\partial^2 u}{\partial z_1\partial \overline{z}_1}=-\frac{\partial^2 u}{\partial z_2\partial \overline{z}_2},$$

we arrive at the equation

$$\left(1-|z_2|^2\right)^2\frac{\partial u}{\partial z_2\partial \overline{z}_2}-\left(1-|z_1|^2\right)^2\frac{\partial u}{\partial z_2\partial \overline{z}_2}=\frac{\partial u}{\partial z_2\partial \overline{z}_2}\left(|z_1|^2-|z_2|^2\right)\left(2-|z_1|^2-|z_2|^2\right)=0 \ (4).$$

From equation (4) follow the relations $\frac{\partial u}{\partial z_2\partial \overline{z}_2}=0$ or $|z_1|=|z_2|$. This means that the function $u(z)$ satisfies the conditions $\frac{\partial u}{\partial z_1\partial \overline{z}_1}=0$ and $\frac{\partial u}{\partial z_2\partial \overline{z}_2}=0$ outside the line $|z_1|=|z_2|$ and $u\in C^2(U^2)$, i.e. it is an n harmonic function in U^2.

If in the previous theorem we consider U^2 (n>2) instead of U^2 , then the question remains open.

Statement 1.

Let U^n be a unit polycircle in $\mathbb{C}^n$ and $u(z)$ and satisfy the following conditions:

function $u(z)$ is M-harmonic in U^n;

function $\overline{u}(z)$ is M-harmonic in U^n.

Then $|u(z)|^2$ is M-subharmonic in U^n.

Statement 2.

Let U^n be a unit polycircle in $\mathbb{C}^n$ and $u(z)$ and satisfy the following conditions:

functions $u(z)$ and $\overline{u}(z)$ are M-harmonic in U^n;

function $|u(z)|^2$ is M-harmonic in U^n.

Then $u(z)=const$ is in U^n.

We do not prove these two statements; the proofs follow easily from Definition 1.3.

CONCLUSION

This article is devoted to some properties of harmonic and power harmonic functions. The main results of the work are the following.

Theorem 2.1.

Let B be the unit ball in $\mathbb{C}^n$ and $u(z)$ and satisfy the following conditions:

a) The function $u(z)$ is harmonic in B;

b) The function $u^m(z)$ is M-harmonic in B.

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Then $u^m(z)$ is pluriharmonic in B.

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Source of support: Nil; **Conflict of interest:** Nil.

Cite this article as:

Madrakhimov, R. and Omonov, O. "Some Properties of Harmonic and Power M-Harmonic Functions." *Sarcouncil Journal of Applied Sciences* 5.3 (2025): pp 1-7

Publisher: SARC Publisher